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Large-Amplitude Free Flexural Vibrations of Laminated Composite Curved Panels Using Shear-Flexible Shell Element

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ABSTRACT

Using C° continuous, QUAD-8 shear-flexible shell element, based on field consistency principle, the nonlinear free flexural vibrations of anisotropic laminated curved panels are studied. The formulation includes transverse shear deformation, in-plane and rotary inertia effects and geometrical nonlinearity. The element is employed to study the large amplitude dynamic behaviour of cylindrical and spherical shells. The frequency versus amplitude curves are obtained from the dynamic response history. The nonlinear governing equations are solved using Wilson-θ numerical integration scheme with θ = 1.4. For each time step, modified Newton-Raphson iterations are employed to achieve equilibrium at the end of that time step. Detailed numerical results are presented, showing the effects of thickness, lamination scheme, material properties and boundary conditions, on nonlinear behaviour.

1. INTRODUCTION

The nonlinear flexural vibration of isotropic cylindrical panel was examined by Reissner[1] and was followed by several investigators[2-4]. Similar studies were made for spherical shells by Hui[5]. The analytical solution for the problem of large deflection dynamic analysis of anisotropic laminated cylindrical panels and spherical shells had been dealt with sparsely in the literature[6,7]. In these few studies[6,7] single mode approach along with perturbation method is employed. To the author's knowledge, solutions based on finite element method have not appeared in the literature for the above investigations.

The geometrical nonlinearity based on Von-Karman strain-displacement relations are considered here. The finite element formulation includes incremental matrices[8] for the nonlinear representation. Shear-flexible, field-consistent shell element[9] QUAD-8 is used to analyse the large amplitude vibration of laminated anisotropic cylindrical panels and doubly curved spherical shells. The frequencies are obtained from the dynamic response history using Wilson-θ method[10]. The results are plotted, showing the effect of material properties, lamination scheme, side-to-thickness ratio and boundary conditions on nonlinear behaviour.

2. FORMULATION

A doubly curved laminated composite shell is considered with the coordinates x, y along the in-plane directions and z along the radial/thickness direction. Using Mindlin formulation, the displacements u, v, w at a point (x,y,z) from the median surface are expressed as functions of mid-plane displacements u_0, v_0 and w, and independent rotations θ_x and θ_y of the normal in xz and yz planes respectively, as

$$u(x,y,z,t) = u_0(x,y,t) + z\theta_x(x,y,t)$$
$$v(x,y,z,t) = v_0(x,y,t) + z\theta_y(x,y,t)$$
$$w(x,y,z,t) = w(x,y,t) \tag{1}$$

Von-Karman's assumptions for moderately large deformation analysis allow Green's strains to be written in terms of mid-plane deformation of Eqn (1) for a shell, based on Novozhilov's theory, as

$$\{\varepsilon\} = \{\varepsilon^{L}\} + \{\varepsilon^{NL}\} \tag{2}$$

where $\{\varepsilon^{L}\} = \begin{Bmatrix} \varepsilon_p^L \\ 0 \end{Bmatrix} + \begin{Bmatrix} Z\varepsilon_b \\ \varepsilon_s \end{Bmatrix}$ and

$$\{\varepsilon^{NL}\} = \begin{Bmatrix} \varepsilon_p^{NL} \\ 0 \end{Bmatrix} \tag{3}$$

The mid-plane strains ε_p^L, bending strain ε_b, and shear strain ε_s in Eqn (3) are written as

$$\{\varepsilon_p^L\} = \begin{Bmatrix} u_{0,x} + (w/R_x) \\ v_{0,y} + (w/R_y) \\ u_{0,y} + v_{0,x} + (2w/R_{xy}) \end{Bmatrix} \tag{3a}$$

$$\{\varepsilon_b\} = -\begin{Bmatrix} \theta_{x,x} \\ \theta_{y,y} \\ \theta_{x,y} + \theta_{y,x} - (u_{o,y}/R_x) - (v_{o,x}/R_y) \end{Bmatrix} \tag{3b}$$

$$\{\varepsilon_s\} = \begin{Bmatrix} \theta_x - w_{,x} + (u_0/R_x) + (v_0/R_{xy}) \\ \theta_y - w_{,y} + (v_0/R_y) + (u_0/R_{xy}) \end{Bmatrix} \tag{3c}$$

where R_x, R_y and R_{xy} are the usual radii of curvature The nonlinear components of in-plane strains are

$$\{\varepsilon_p^{NL}\} = \begin{Bmatrix} (1/2)\, w_{,x}^2 \\ (1/2)\, w_{,y}^2 \\ w_{,x} w_{,y} \end{Bmatrix} \tag{3d}$$

If $\{N\}$ represents the membrane stress resultants (N_{xx}, N_{yy}, N_{xy}) and $\{M\}$ the bending stress resultants (M_{xx}, M_{yy}, M_{xy}), one can relate N and M to membrane strains $\{\varepsilon_p\}$ i.e. $\{\varepsilon_p^L\} + \{\varepsilon_p^{NL}\}$ and bending strains $\{\varepsilon_b\}$ through the constitutive relations, as

$$\{N\} = [A_{ij}]\{\varepsilon_p\} + [B_{ij}]\{\varepsilon_b\} \text{ and } \{M\} = [B_{ij}]\{\varepsilon_p\} + [D_{ij}]\{\varepsilon_b\} \tag{4}$$

where A_{ij}, D_{ij} and B_{ij} $(i, j = 1,2,3)$ are extensional, bending and bending-extensional stiffness coefficients of the composite laminate. Similarly, the transverse shear force $\{Q\}$ representing the quantities $\{Q_{xz}, Q_{yz}\}$ are related to the transverse shear strains $\{\varepsilon_s\}$ through the constitutive relations as

$$\{Q\} = [E_{ij}]\{\varepsilon_s\} \tag{5}$$

where $E_{ij}(i, j = 4,5)$ are the transverse shear stiffness coefficients of the laminate.

For a composite laminate of thickness h, consisting of N layers with stacking angles $\phi_i = (i = 1, \ldots, N)$ and layer thickness h_i $(i = 1, \ldots, N)$, the necessary expressions to compute the stiffness coefficients, available in the literature[11], are used here. The potential energy functional U is given by

$$U(\delta) = \frac{1}{2}\int_A \Big[\{\varepsilon_p\}^T[A_{ij}]\{\varepsilon_p\} + \{\varepsilon_p\}^T[B_{ij}]\{\varepsilon_b\} + \{\varepsilon_b\}^T[B_{ij}]\{\varepsilon_p\} + \{\varepsilon_b\}^T[D_{ij}]\{\varepsilon_b\} + \{\varepsilon_s\}^T[E_{ij}]\{\varepsilon_s\}\Big]\, dA \tag{6}$$

where δ is the vectors of degrees of freedom.

Following the incremental procedure[8] the strain energy functional U is rewritten as

$$U(\delta) = \{\delta\}^T\Big[(1/2)[K] + (1/6)[N1] + (1/12)[N2] + (1/2)[N3]\Big]\{\delta\} \tag{7}$$

where $[K]$ is linear stiffness matrix, and $[N1]$ $[N2]$ are nonlinear stiffness matrices, and $[N3]$ is the shear stiffness matrix.

The kinetic energy of the shell is given by

$$T(\delta) = (1/2)\int_A \Big[p(\dot{u}^2 + \dot{v}^2 + \dot{w}^2) + I(\dot{\theta}_x^2 + \dot{\theta}_y^2)\Big]\, dA \tag{8}$$

where $p = \int_0^h \rho dz$, $I = \int_0^h z^2 \rho dz$ and ρ is the mass density.

Substituting Eqns (7) and (8) in Lagrange's equation of motion, one obtains the governing equation for the free flexural vibration of the shell as

$$\Big[[K] + (1/2)[N1] + (1/3)[N2] + [N3]\Big]\{\delta\} + [M]\{\ddot{\delta}\} = \{0\} \tag{9}$$

where $[M]$ is the mass matrix.

Equation (9) is solved using the implicit method[10].

In the implicit method, equilibrium conditions are considered at the same time step for which solution is sought. If the solution is known at time t and one wishes to obtain the displacements etc., at time $t+\Delta t$, then equilibrium equations considered at time $t+\Delta t$ are given as

$$\Big[[N(\delta)]\{\delta\}\Big]_{t+\Delta t} + [M]\{\ddot{\delta}\}_{t+\Delta t} = \{0\} \tag{10}$$

where $[M]$ is the mass matrix, $\{\delta\}_{t+\Delta t}$ / $\{\ddot{\delta}\}_{t+\Delta t}$ are the vectors of nodal displacements and accelerations at time

$t+\Delta t$ respectively, $[[N(\delta)]\ \{\delta\}]_{t+\Delta t}$ is the internal force vector at time $t+\Delta t$, and is given as

$$\left[[N(\delta)]\{\delta\}\right]_{t+\Delta t} = \left\{\left[[K] + (1/2)[N1] + (1/3)[N2] + [N3]\right]\{\delta\}\right\}_{t+\Delta t} \tag{11}$$

In developing equations for the implicit integration, the internal forces $[N(\delta)]\ \{\delta\}$ at time $t+\Delta t$ is written in terms of internal forces at time t, using tangent stiffness approach, as

$$\left[[N(\delta)]\ \{\delta\}\right]_{t+\Delta t} = \left[[N(\delta)]\ \{\delta\}\right]_t + [K_T(\delta)]_t\{\Delta\delta\} \tag{12}$$

where $[K_T(\delta)] = \left[[K] + [N1] + [N2] + [N3]\right]$ is the tangential stiffness matrix and $\{\Delta\delta\} = \{\delta\}_{t+\Delta t} - \{\delta\}_t$.

Substituting Eqn (12) into Eqn (10), one obtains the governing equations at $t+\Delta t$ as

$$[M]\ \{\ddot{\delta}\}_{t+\Delta t} + [K_T(\delta)]_t\ \{\Delta\delta\} = -[N(\delta)]\ \{\delta\}_t \tag{13}$$

To improve the solution accuracy and to avoid the numerical instabilities, it is necessary to employ iteration within each time step to achieve equilibrium.

The resulting nonlinear equations obtained by the above procedure are solved by Wilson-θ numerical integration method. Equilibrium is achieved for each time step through modified Newton-Raphson iteration until the convergence criteria[12] (modified absolute norm, modified Euclidean norm and maximum norm) are satisfied within the specific tolerance limit of less than 1 per cent.

3. ELEMENT DESCRIPTION

The laminated shell element employed here is a C^0 continuous shear flexible element and needs five nodal degrees of freedom u, v, w, θ_x and θ_y at eight nodes in QUAD-8 element, shown in Fig. 1.

If the interpolation functions for QUAD-8 are used directly to interpolate the five field variables u to θ_y in deriving the shear strains and membrane strains, the element will lock and show oscillations in the shear and membrane stresses. Field consistency requires that the

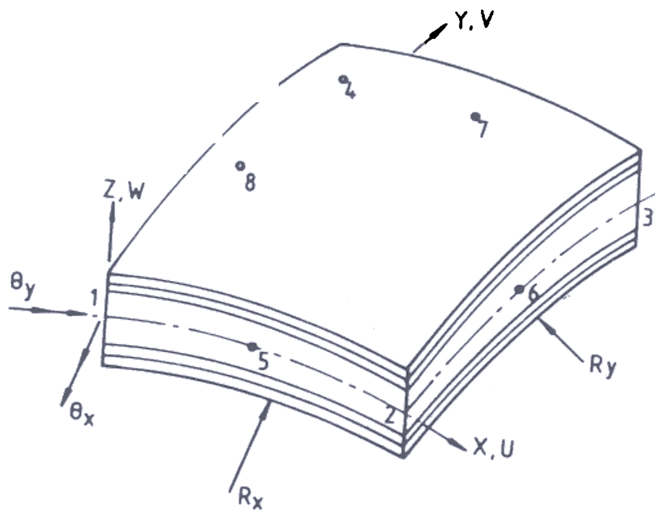

Figure 1 Geometry of laminated shell element (QUAD-8).

transverse shear strains and membrane strains must be interpolated in a consistent manner. Thus u, v, θ_x and θ_y terms in the expressions for $\{\varepsilon_s\}$ given in Eqn (3c) have to be consistent with field functions $w_{,x}$ and $w_{,y}$[9]. Similarly, w term in the expressions of $\{\varepsilon_p\}$ given in Eqn (3a) has to be consistent with field functions ($u_{,x}$, $v_{,y}$) and ($u_{,y}$ $v_{,x}$). This is achieved by using field redistributed substitute shape functions to interpolate those specific terms which must be consistent[9].

4. RESULTS AND DISCUSSION

In the present study, the eight-noded isoparametric field-consistent shell element is employed. Since the element is derived from the field consistency approach, exact integration is performed to evaluate all the strain energy terms. Based on progressive mesh refinement, a 4 × 4 mesh is found to be adequate to model one quadrant of the shell for analysis. The initial conditions for the nonlinear free vibration study are zero values for displacements/rotations and non-zero values for velocities. The initial velocity vectors are proportional to the normalised linear flexural mode vectors. The response curves are obtained by varying the initial velocity vectors. The frequency and the corresponding amplitude are evaluated from the response curves. The shear correction factor is taken as 5/6. The value of θ in Wilson-θ method is assumed as 1.4 which corresponds to unconditionally stable scheme in linear analysis.

Since no estimate of the time step for the nonlinear analysis is available in the literature, the critical time step of a conditionally stable finite-difference scheme[13,14] is introduced as a guide and a convergence study is conducted to select a time step which yields a stable and accurate solution. The critical time steps given for thin and moderately thick plates[13,14] are

$$\Delta t \leq 0.25\ (\rho\ h/D)^{1/2}\ \Delta x \tag{14}$$

$$\Delta t \leq [\{\rho(1-\upsilon^2)/E\}/\{2 + (1-\upsilon)\ \pi^2/12$$

$$(1 + 5\ \Delta x^2/h^2\ )\}]^{1/2}\ \Delta x \tag{15}$$

where Δx is the minimum distance between the element mode points and D, E are the flexural rigidity and Young's modulus respectively. υ is the Poisson's ratio.

Due to anisotropic nature of composite plates/shells, even though the geometry and loading are symmetric about the axes, one has to be careful in assuming biaxial symmetry for the analysis. In the present study, after verifying the results of quarter plate/shell idealisation with those of full plate/shell idealisation, the following boundary conditions are used:

Simply supported $u = w = \theta_y = 0$ on $x = -a, a$

$w = \theta_x = 0$ on $y = -b, b$

Clamped support $u = v = w = \theta_x = \theta_y = 0$ on $x = -a, a$ and $y = -b, b$

Line of symmetry :

Cross-ply : $v = \theta_y = 0$ on $y = 0$, $u = \theta_x = 0$ on $x = 0$

Angle-ply : $u = \theta_y = 0$ on $y = 0$, $v = \theta_x = 0$ on $y = 0$

To start with, the present formulation is validated by considering the linear vibration of a simply supported cross-ply doubly curved spherical shells and nonlinear vibration of isotropic plates. The results obtained are compared in Tables 1 & 2 with those available in literature[15,16]. The two are found to be in very good agreement.

Numerical calculation is carried out for two types of materials[7] whose elastic constants are as follows :

$E_1/E_2 = 7.6$, $G_{12}/E_2 = 0.4$, $G_{23}/E_2 = G_{13}/E_2$ 0.2, $\upsilon_{12} = 0.3$

2 $E_1/E_2 = 2.0$, $G_{12}/E_2 = 0.4$, $G_{23}/E_2 = G_{13}/E_2 = 0.2$

$\upsilon_{12} = 0.3$

Table 1. Nondimensional frequencies $\bar{\omega} = \omega\ a^2\ (\rho/E_2h)^{1/2}$ of simply supported 0°/90°/90°/0° spherical shell

R/a	$a/h = 100$ Present study	Ref. [15]	$a/h = 10$ Present study	Ref. [15]
	126.2850	126.330	16.153	16.172
∠	68.2790	68.294	13.419	13.477
4	37.0730	37.082	12.525	12.552
10	20.383	20.380	12.275	12.280
10^{30}	15.1933	15.184	12.230	12.226

($E_1/E_2 = 25$, $G_{12} = G_{13} = 0.5\ E_2$, $G_{23} = 0.2\ E_2$, $\upsilon_{12} = 0.25$, $\rho = 1.0$, $a/b = 1$, $R_1 = R_2 = R$)

Table 2. Frequency ratio (ω_{NL}/ω_L)* of nonlinear vibrations of isotropic simply supported square plates

h/a	w/h	Ref. 16	Present study
	0.2	1.02599	1.02504
	0.4	1.10027	1.10020
0.001	0.6	1.21402	1.20803
	0.8	1.35735	1.35074
	1.0	1.52192	1.51347

ω_{NL} nonlinear frequency; ω_L, linear frequency

Here, E_1 and E_2 are Young's moduli along the longitudinal and transverse directions of the fibre respectively. All the layers are of equal thickness.

The amplitude-frequency relations are shown in Figs 2 & 3 respectively for the simply supported and clamped symmetrically layered cross-ply cylindrical panel (0°/90°/0°/90°/0°, $a/b = 1$, $a/h = 100$, $R/a = 10$) along with the existing analytical solutions. For low values of amplitude, the nonlinear behaviour predicted by the present model agrees well with that seen in Ref. 7 whereas there is a discrepancy in the results at higher amplitudes. It is worthwhile to mention here that the formulation given in Ref. 7 is based on classical shell theory and neglects the in-plane and rotary inertia effects. The present study also confirms that nonlinearity is of softening type (frequency ratio decreases with increasing amplitude) for thin panels, irrespective of support conditions and material properties.

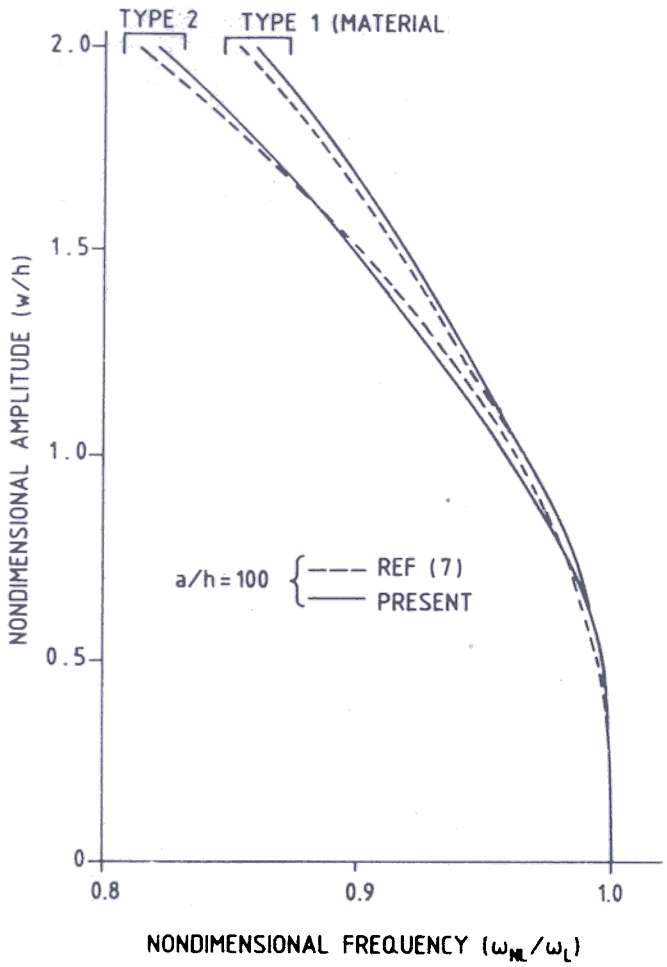

Figure 2. **Influence of large amplitude on frequency for simply supported cross-ply cylindrical panel (0°/90°/0°/90°/0°).**

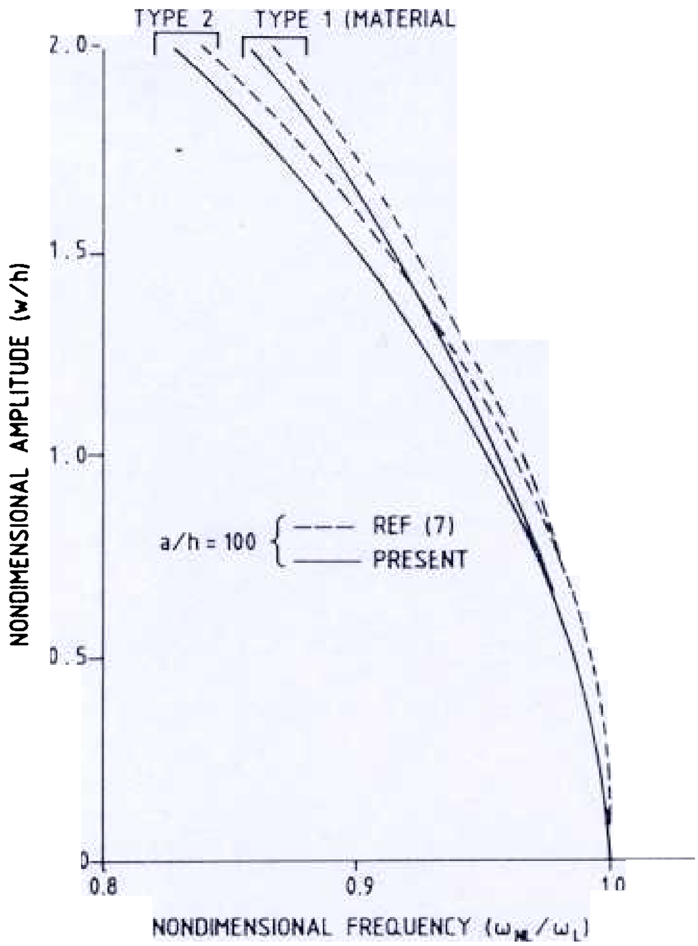

Figure 3. Influence of large amplitude on frequency for clamped cross-ply cylindrical panel (0°/90°/0°/90°/0°).

The effects of two-layered simply supported cross-ply as well as angle-ply laminates on nonlinear behaviour are presented in Figs 4 & 5 for cylindrical panels and spherical shells respectively. The results are presented for two values of thickness parameter (a/h = 10, 100; material Type 1). It is observed that nonlinearity in the case of moderately thick shells is of hardening type (frequency ratio increases with amplitude), whether it is for cylindrical panels or doubly curved spherical shells, whereas it is of softening type for thin panels.

5. SUMMARY

The investigation on the nonlinear free flexural vibrations of cylindrical and spherical shells has been carried out here, for the first time, using a shear-flexible shell element along with the dynamic response approach. The nonlinear behaviour of thin and moderately thick shells is brought out from the free vibration response.

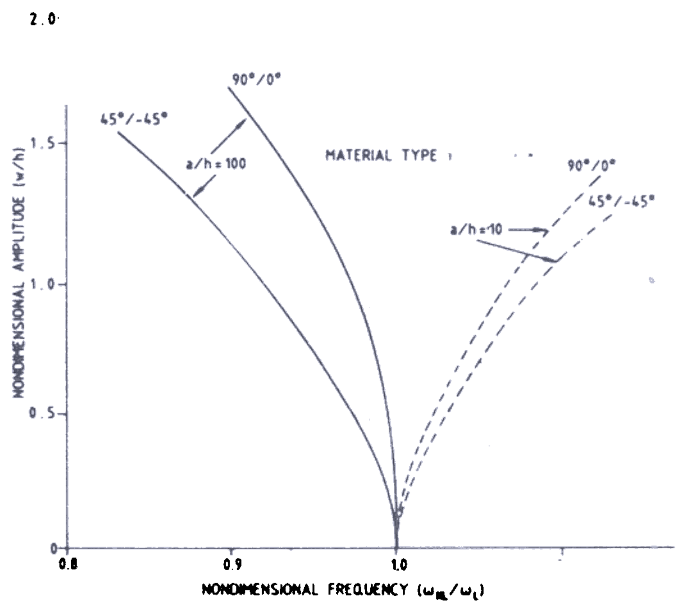

Figure 4. Influence of large amplitude on frequency for simply supported two-layered cylindrical panel.

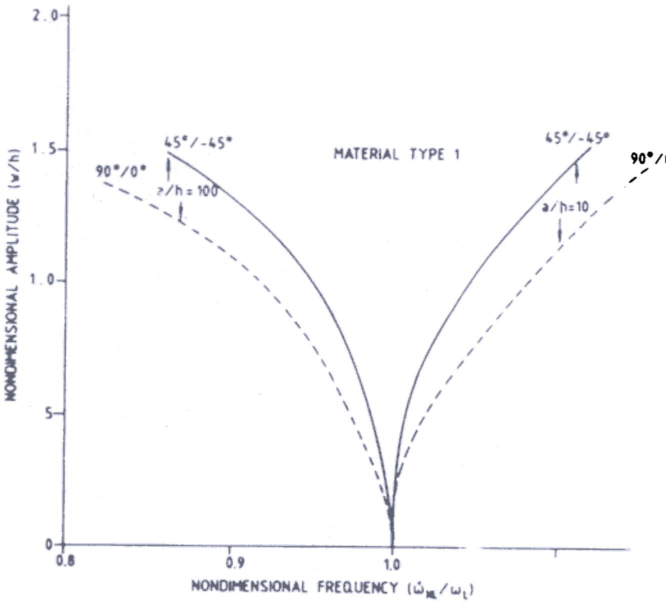

Figure 5. Influence of large amplitude on frequency for simply supported two-layered doubly curved spherical shell.

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